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Controlling the bias of robust small-area estimators

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SUMMARY

Sinha & Rao (2009) proposed estimation procedures designed for small-area means, based on robustified maximum likelihood estimators and robust empirical best linear unbiased predictors. Their methods are of the plug-in type and may be biased. Bias-corrected estimators have been proposed by Chambers et al. (2013). Here, we investigate two new approaches: one relying on the work of Chambers (1986), and the second using the concept of conditional bias to measure the influence of units in the population. These two classes of estimators also include correction terms for the bias but are both fully bias-corrected, in the sense that the corrections account for the potential impact of the other domains on the small area of interest. Monte Carlo simulations suggest that the Sinha–Rao method and the bias-adjusted estimator of Chambers et al. (2013) may exhibit a large bias, while the new procedures often offer lower bias and mean squared error. A parametric bootstrap procedure is considered for constructing confidence intervals.

Some key words: Conditional bias; Corrected-bias estimator; Influence measure; Model-based inference; Outlier; Small-area estimation; Survey sampling.

1. INTRODUCTION

In the last two decades, the demand for small-area estimators has been growing in most countries. This has led survey statisticians to develop theoretically sound and yet practical estimation procedures, providing reliable estimators for the variables of interest at the small-area level (e.g., Rao, 2003). Small areas are defined as domains whose sample sizes are not large enough to justify the use of direct estimators. Typically, the information in the domain-specific sample data does not suffice to obtain efficient estimators. For example, direct estimators, such as the classical Horvitz–Thompson estimator, are not appropriate in such circumstances. To overcome these small sample size problems, small-area estimation techniques have been based on model-based methods and the use of auxiliary information. Consequently, so-called indirect estimators have been proposed, typically relying on linear models, and observed values from related areas are used in order to predict the small-area characteristics. The construction of indirect estimators requires auxiliary information at the area and/or unit levels, and efficiency requires a correct specification of the model linking the study variable y to the auxiliary information. See Rao (2003, 2005) for comprehensive accounts of small-area estimation.

Let U be a finite population of size N, which is partitioned into k subpopulations $U_1, \ldots, U_k$, of sizes $N_1, \ldots, N_k$, respectively. Thus $U = \bigcup_{i=1}^{k} U_i$ such that $U_i \cap U_l = \emptyset$, $i \neq l$, and $N = \sum_{i=1}^{k} N_i$. The domain sizes N_i are assumed to be known. For a variable of interest y, let y_{ij} be the value of y attached to the jth element of the ith area. The parameters of interest are the small-area means $\theta_i = N_i^{-1} \sum_{j \in U_i} y_{ij}$, $(i = 1, \ldots, k)$.

A sample s of size n is selected from U according to a given sampling plan $p(s)$. Let $s_i = s \cap U_i$ denote the ith area specific sample of size n_i. Thus, $s = \bigcup_{i=1}^{k} s_i$ and $n = \sum_{i=1}^{k} n_i$. Let x_{ij} be a deterministic vector of dimension p, containing the auxiliary variables corresponding to unit j of the area i. The ith area-specific totals, $\sum_{j \in U_i} x_{ij}$, are also supposed to be available.

In this paper, we focus on the basic unit-level model that can be expressed as

$$y_{ij} = x_{ij}^{\mathrm{T}} \beta + v_i + e_{ij} \quad (j = 1, \ldots, n_i), \tag{1}$$

for the ith area, where v_i denotes a random variable associated with the small area i. The use of a random effects model allows us to account for the between-area variation that is not explained by the auxiliary information. The error term is e_{ij}, and it is assumed that e_{ij} and v_i are independent random variables for all i and j. Classical distributional assumptions for the error terms and the random effects rely on normal theory, assuming that e_{ij} are independent $\mathcal{N}(0, \sigma_e^2)$, and v_i are independent and identically distributed $\mathcal{N}(0, \sigma_v^2)$ random variables, respectively. We collect the variance components in the vector $\delta = (\sigma_e^2, \sigma_v^2)^{\mathrm{T}}$. Model (1) is a special case of the more general linear mixed model. In matrix notation, the unit-level model (1) can be compactly rewritten as

$$y_i = X_i \beta + v_i 1_{n_i} + e_i \quad (i = 1, \ldots, k),$$

where $X_i = (x_{i1}, \ldots, x_{in_i})^{\mathrm{T}}$ is a matrix of dimension $n_i \times p$ and 1_{n_i} corresponds to a vector of ones of dimension $n_i \times 1$. The classical normality assumption for the error term is $e_i \sim \mathcal{N}_{n_i}(0, \sigma_e^2 I_{n_i})$, with $e_i = (e_{i1}, \ldots, e_{in_i})^{\mathrm{T}}$, where I_{n_i} represents the identity matrix of order n_i. The variance-covariance matrix of $y_i = (y_{i1}, \ldots, y_{in_i})^{\mathrm{T}}$ is thus $V_i(\delta) = \sigma_e^2 I_{n_i} + \sigma_v^2 1_{n_i} 1_{n_i}^{\mathrm{T}}$. Popular estimation methods for δ are maximum likelihood or restricted maximum likelihood estimators that in turn can be used to obtain empirical best linear unbiased estimators of β and empirical best linear unbiased predictors of v_i. See § 2.

Outliers occur frequently in surveys, especially in business surveys, since economic variables whose distributions are highly skewed are typically studied. Usually, the distributions of such variables display heavy right tails, and outlying units present unusual large values for such variables of interest. In an ideal survey sampling set-up, the survey design accounts for such large units. However, in applications, imperfect auxiliary information when stratifying the population may yield outliers in the sample. In the terminology of Chambers (1986), these outliers are representative, in the sense that they are representative of the nonsampled part of the population. See Lee (1995), Duchesne (1999) and Beaumont & Rivest (2009) for reviews of outliers in survey sampling. In a small-area framework, empirical best linear unbiased predictors are efficient under correct model specification and distributional assumptions, but they may be highly sensitive to the presence of outliers. Including or excluding outlying units in the calculation of the empirical best linear unbiased predictor may have a large impact on its magnitude. See, e.g., Fellner (1986), Stahel & Welsh (1997) and Sinha & Rao (2009).

Robust small-area estimation has received considerable attention in recent years. Chambers & Tzavidis (2006) proposed robust estimators based on the so-called M-quantile modelling approach of the conditional distribution of the variable of interest given the auxiliary information. Ghosh et al. (2008) studied robust procedures using Bayesian methods. Tzavidis et al. (2010) studied robust prediction of small-area means and distributions. They proposed a general framework for robust small-area estimation, based on representing the small-area estimator as a functional of a predictor of this small-area cumulative distribution function. Sinha & Rao (2009) studied the robustness of the empirical best linear unbiased predictor and proposed resistant methods for small-area estimation. They estimated the mean squared errors of the robust estimators of the small-area means, using a parametric bootstrap procedure. Chambers et al. (2013) studied robust small-area estimation, and noted that the methods of Sinha & Rao (2009) are based on plug-in robust versions of the original empirical best linear unbiased predictors. Work of Chambers (1986) suggests that these approaches may introduce nonnegligible prediction biases, and it appears necessary to include additional terms to correct for the bias. Chambers et al. (2013) proposed bias-corrected predictive estimators that can be less biased than the estimators of Sinha & Rao (2009). Chambers et al. (2011) also derived analytical mean squared error estimators for outlier robust predictors of small-area means.

The main objective here is to propose new robust estimators for small-area means with correction terms for the bias. Two approaches are studied. Under the first approach, the arguments of Chambers (1986) are adapted to the unit-level model. In the second approach, the concept of conditional bias is used. Muñoz-Pichardo et al. (1995) proposed to study the influence of a given observation in general linear models using the conditional bias. In Moreno-Rebollo et al. (1999), the concept of conditional bias is proposed as a measure of influence in the context of design-based inference. Beaumont et al. (2013) advocated its use in a model-based framework, proposed robust estimators for population means and totals, and established links with the robust methods of Chambers (1986) in classical linear models. Both classes of predictors can be interpreted as a compromise between the Sinha–Rao method and the nonrobust empirical best linear unbiased predictors, through ψ-functions and tuning constants.

2. Preliminaries

Several robust estimation techniques have been developed for estimating the unit-level model (1). First, the empirical best linear unbiased predictor method finds explicit expressions for the best linear unbiased estimator and best linear unbiased predictors of β and $v_1, \ldots, v_k$, respectively, assuming δ known. They are given by

$$\tilde{\beta}(\delta) = \left(\sum_{h=1}^{k} X_h^{\mathrm{T}} V_h^{-1} X_h \right)^{-1} \sum_{h=1}^{k} X_h^{\mathrm{T}} V_h^{-1} y_h,$$

and $\tilde{v}_h(\delta) = \sigma_v^2 1_{n_h}^{\mathrm{T}} V_h^{-1}(y_h - X_h \tilde{\beta})$ with $V_h \equiv V_h(\delta)$ and $\tilde{\beta} \equiv \tilde{\beta}(\delta)$. These expressions can be justified using a penalized-likelihood criterion, and they are derived in Rao (2003, Ch. 6). The best linear unbiased predictor of θ_i can be written as

$$\hat{\theta}_i(\delta) = N_i^{-1} \left[\sum_{j \in s_i} y_{ij} + \sum_{j \in U_i - s_i} \{ x_{ij}^{\mathrm{T}} \tilde{\beta}(\delta) + \tilde{v}_i(\delta) \} \right]. \quad (2)$$

It can be shown that the predictor (2) minimizes the model mean squared error under model (1) in the class of linear model unbiased predictors of θ_i (Rao, 2003, p. 98). Using maximum likelihood or restricted maximum likelihood techniques for δ, a suitable estimator $\hat{\delta}$ is inserted into the best linear unbiased estimator and best linear unbiased predictors. This two-stage method leads to the empirical best linear unbiased estimator of β and the empirical best linear unbiased predictors of

v_h, that is $\hat{\beta} = \tilde{\beta}(\hat{\delta})$ and $\hat{v}_h = \tilde{v}_h(\hat{\delta})$. The empirical best linear unbiased predictor of θ_i is thus

$$\hat{\theta}_{i\,\mathrm{EBLUP}} = \hat{\theta}_i(\hat{\delta}). \tag{3}$$

Since the best linear unbiased predictor and its empirical counterpart are sensitive to the presence of outliers, robust versions have been proposed by Fellner (1986). Robust estimators and predictors of β and $v_1, \ldots, v_k$ are developed in robustifying the model equations leading to the best linear unbiased estimator and best linear unbiased predictors, for known δ. The so-called Fellner equations are

$$\sigma_e^{-1} \sum_{h=1}^{k} X_h^{\mathrm{T}} \Psi\{\sigma_e^{-1}(y_h - X_h\beta - v_h 1_{n_h})\} = 0,$$

$$\sigma_e^{-1} 1_{n_i}^{\mathrm{T}} \Psi\{\sigma_e^{-1}(y_i - X_i\beta - v_i 1_{n_i})\} - \sigma_v^{-1}\psi_b(\sigma_v^{-1} v_i) = 0 \quad (i = 1, \ldots, k). \tag{4}$$

The Ψ function is an $n_i \times 1$ vector defined as $\Psi(u) = \{\psi_b(u_1), \psi_b(u_2), \ldots, \psi_b(u_{n_i})\}^{\mathrm{T}}$, with

$$\psi_b(u) = \min\{|b|, \max(-|b|, u)\}. \tag{5}$$

In a classical robust estimation framework, the choice of the tuning constant b is often dictated by efficiency considerations in a perfectly observed normal model. For example, a popular choice is $b = 1{\cdot}345$. Then, using robustified Henderson equations (Rao, 2003 and Sinha & Rao, 2009), robust estimators of the variance components δ are obtained.

Sinha & Rao (2009) used an alternative two-step approach for constructing robust estimators, based on robustified score equations of the Gaussian maximum likelihood estimators. Assuming normality, a Gaussian multivariate model can be formulated as a function of $(\beta^{\mathrm{T}}, \delta^{\mathrm{T}})^{\mathrm{T}}$. The maximum likelihood estimators are then solutions of the score equations

$$\sum_{h=1}^{k} X_h^{\mathrm{T}} V_h^{-1}(y_h - X_h\beta) = 0, \tag{6}$$

$$\sum_{h=1}^{k} \left\{ (y_h - X_h\beta)^{\mathrm{T}} V_h^{-1} \frac{\partial V_h}{\partial \delta_l} V_h^{-1}(y_h - X_h\beta) - \mathrm{tr}\left(V_h^{-1} \frac{\partial V_h}{\partial \delta_l} \right) \right\} = 0 \quad (l = 1, \ldots, k). \tag{7}$$

Since the maximum likelihood estimators are not robust, (6) and (7) need to be robustified. One proposal has been considered in Huggins (1993); see also Richardson & Welsh (1995). Sinha & Rao (2009) proposed the robustified equations

$$\sum_{h=1}^{k} X_h^{\mathrm{T}} V_h^{-1} U_h^{1/2} \Psi(r_h) = 0, \tag{8}$$

$$\sum_{h=1}^{k} \left\{ \Psi^{\mathrm{T}}(r_h) U_h^{1/2} V_h^{-1} \frac{\partial V_h}{\partial \delta_l} V_h^{-1} U_h^{1/2} \Psi(r_h) - \mathrm{tr}\left(K_h V_h^{-1} \frac{\partial V_h}{\partial \delta_l} \right) \right\} = 0 \quad (l = 1, \ldots, k), \tag{9}$$

where $r_h = U_h^{-1/2}(y_h - X_h\beta)$, $U_h = \mathrm{diag}(V_h)$, $K_h = E\{\psi_b^2(r)\} I_{n_h}$, and r is a standard normal random variable. Solving equations (8) and (9) usually requires iterative algorithms. Let $\hat{\beta}_R$ and $\hat{\delta}_R = (\hat{\sigma}_{eR}^2, \hat{\sigma}_{vR}^2)^{\mathrm{T}}$ be the robust estimators of β and δ, obtained by solving (8) and (9). In the second step, robust predictors of v_i are obtained by solving (4), conditionally given $(\hat{\beta}_R^{\mathrm{T}}, \hat{\delta}_R^{\mathrm{T}})^{\mathrm{T}}$.

We denote these predictors by $\hat{v}_{iR} \equiv \hat{v}_{iR}(\hat{\delta}_R)$. Having robustly estimated all the parameters, the Sinha–Rao robust empirical best linear unbiased predictor of the plug-in type for θ_i can be written as

$$\hat{\theta}_{i\mathrm{SR}} = N_i^{-1} \sum_{j \in s_i} y_{ij} + N_i^{-1} \sum_{j \in U_i - s_i} x_{ij}^{\mathrm{T}} \hat{\beta}_R + \left(1 - n_i N_i^{-1}\right) \hat{v}_{iR}. \tag{10}$$

Chambers et al. (2013) argued that $\hat{\theta}_{i\mathrm{SR}}$ in (10) may involve a large prediction bias when the population data are drawn from a mixture distribution, for which the means of the outliers and non-outliers are different. They proposed a bias-corrected version of the Sinha–Rao robust empirical best linear unbiased predictor, using an approach similar to that advocated by Welsh & Ronchetti (1998) for robust prediction of the empirical distribution function of y values in area i. An estimator of the area mean θ_i is then defined by the value of the mean functional

$$\hat{\theta}_{i\mathrm{CCST}} = \hat{\theta}_{i\mathrm{SR}} + \left(n_i^{-1} - N_i^{-1}\right) \sum_{j \in s_i} \phi_i \psi_c \left(\frac{y_{ij} - x_{ij}^{\mathrm{T}} \hat{\beta}_R - \hat{v}_{iR}}{\phi_i} \right), \tag{11}$$

where the weights ϕ_i are the median absolute deviation of the ith area residuals, $y_{ij} - x_{ij}^{\mathrm{T}} \hat{\beta}_R - \hat{v}_{iR}$, and $\psi_c(t)$ is given by (5). When the tuning constant $c = 0$, the estimator $\hat{\theta}_{i\mathrm{CCST}}$ defined by (11) reduces to the robust estimator $\hat{\theta}_{i\mathrm{SR}}$. Thus, the bias-corrected estimator includes an additional term, whose role consists of controlling the potential bias. The correction factor given by the last term on the right-hand side of (11) is based only on $j \in s_i$. Thus, the bias-corrected robust empirical best linear unbiased predictor of Chambers et al. (2013) neglects the information associated with the units which are not in the area i, and which may still influence the estimators. On the other hand, $\hat{\theta}_{i\mathrm{CCST}}$ should be less biased when $c = \infty$, at the price of being less robust. The estimator (11) with $c = \infty$ does not reduce to the nonrobust predictor of the area mean $\hat{\theta}_{i\mathrm{EBLUP}}$ defined by (3). This is due to the fact that $\hat{\theta}_{i\mathrm{EBLUP}} \neq \hat{\theta}_{i\mathrm{SR}} + (n_i^{-1} - N_i^{-1}) \sum_{j \in s_i} (y_{ij} - x_{ij}^{\mathrm{T}} \hat{\beta}_R - \hat{v}_{iR})$, in general. This issue is also discussed in § 3·1.

3. Fully bias-corrected robust predictors

3·1. *Robust predictor based on Chambers' approach*

The predictor $\hat{\theta}_{i\mathrm{EBLUP}}$ defined by (3) can alternatively be written as

$$\hat{\theta}_{i\mathrm{EBLUP}} = N_i^{-1} \sum_{h=1}^{k} \sum_{j \in s_h} w_{ihj}(\hat{\delta}) y_{hj},$$

where the weights satisfy the relations

$$w_{ihj}(\delta) = \begin{cases} k^{-1} a_i X_h^{\mathrm{T}} C_h^{(j)} & (j \in s_h), \\ 1 + k^{-1} a_i X_i^{\mathrm{T}} C_i^{(j)} + (N_i - n_i) \sigma_v^2 1_{n_i}^{\mathrm{T}} C_i^{(j)} & (j \in s_i), \end{cases} \tag{12}$$

with $a_i = \{\sum_{j \in U_i - s_i} x_{ij}^{\mathrm{T}} - \sigma_v^2 (N_i - n_i) 1_{n_i}^{\mathrm{T}} V_i^{-1}(\delta) X_i\}\{k^{-1} \sum_{i=1}^{k} X_i^{\mathrm{T}} V_i^{-1}(\delta) X_i\}^{-1}$, and $C_i(\delta) = V_i^{-1}(\delta)$ is a matrix satisfying $C_i(\delta) \equiv C_i = (C_i^{(1)}, \ldots, C_i^{(n_i)})$, with $C_i^{(j)}$ corresponding to the jth column of C_i. The weights of sampled units are indexed by ihj, and the form of the weight is different depending if $j \in s_i$ or $j \in s_h$, $h \neq i$. Clearly the observations in the area i of interest should have more weight than the observations in s_h, $h \neq i$. It is easily seen that the weights $w_{ihj}(\hat{\delta})$ $(j \in s_h)$ satisfy the calibration constraints

$$\sum_{h=1}^{k} \sum_{j \in s_h} w_{ihj}(\hat{\delta}) x_{hj}^{\mathrm{T}} = \sum_{j \in U_i} x_{ij}^{\mathrm{T}}. \tag{13}$$

Let $\alpha \in \mathcal{R}^p$ be an arbitrary vector of dimension p and let $u_1, \ldots, u_k$ be arbitrary random variables. Using arguments similar to those of Chambers (1986) and (13), the empirical best linear unbiased predictor defined by (3) can be written as

$$\begin{aligned}
\hat{\theta}_{i\text{EBLUP}} &= N_i^{-1} \sum_{h=1}^{k} \sum_{j \in s_h} w_{ihj}(\hat{\delta})(y_{hj} - x_{hj}^{\mathrm{T}}\alpha - u_h + x_{hj}^{\mathrm{T}}\alpha + u_h), \\
&= N_i^{-1} \sum_{j \in s_i} y_{ij} + N_i^{-1} \sum_{j \in U_i - s_i} (x_{ij}^{\mathrm{T}}\alpha + u_i) + N_i^{-1} \sum_{j \in s_i} \{w_{ij}(\hat{\delta}) - 1\}(y_{ij} - x_{ij}^{\mathrm{T}}\alpha - u_i) \\
&\quad + N_i^{-1} \sum_{\substack{h=1 \\ h \neq i}}^{k} \sum_{j \in s_h} w_{ihj}(\hat{\delta})(y_{hj} - x_{hj}^{\mathrm{T}}\alpha - u_h) + N_i^{-1} \sum_{h=1}^{k} W_{ih}(\hat{\delta}) u_h,
\end{aligned} \tag{14}$$

where the weights $W_{ih}(\hat{\delta})$ are

$$W_{ih}(\hat{\delta}) = \begin{cases} \sum_{j \in s_i} w_{iij}(\hat{\delta}) - N_i & (h = i), \\ \sum_{j \in s_h} w_{ihj}(\hat{\delta}) & (h \neq i). \end{cases}$$

In (14), the vector α is arbitrary. A natural candidate is $\hat{\beta}_R$, solution of equations (8) and (9). Similarly, a suitable predictor for u_h is $\hat{v}_{hR}$, see § 2. The empirical best linear unbiased predictor $\hat{\theta}_{i\text{EBLUP}}$ can be written as

$$\begin{aligned}
\hat{\theta}_{i\text{EBLUP}} = \hat{\theta}_{i\text{SR}} &+ N_i^{-1} \sum_{j \in s_i} \{w_{ij}(\hat{\delta}) - 1\}(y_{ij} - x_{ij}^{\mathrm{T}}\hat{\beta}_R - \hat{v}_{iR}) \\
&+ N_i^{-1} \sum_{\substack{h=1 \\ h \neq i}}^{k} \sum_{j \in s_h} w_{ihj}(\hat{\delta})(y_{hj} - x_{hj}^{\mathrm{T}}\hat{\beta}_R - \hat{v}_{hR}) + N_i^{-1} \sum_{h=1}^{k} W_{ih}(\hat{\delta})\hat{v}_{hR},
\end{aligned} \tag{15}$$

where $\hat{\theta}_{i\text{SR}}$ is given by (10). The last three terms of (15) can be viewed as correction terms for the bias, which may be affected by large residuals. A robust predictor of θ_i is obtained, robustifying the representation (15)

$$\begin{aligned}
\hat{\theta}_{i\text{C}} = \hat{\theta}_{i\text{SR}} &+ N_i^{-1} \sum_{j \in s_i} \psi_{c_1}\{(w_{ij}(\hat{\delta}) - 1)(y_{ij} - x_{ij}^{\mathrm{T}}\hat{\beta}_R - \hat{v}_{iR})\} \\
&+ N_i^{-1} \sum_{\substack{h=1 \\ h \neq i}}^{k} \sum_{j \in s_h} \psi_{c_1}\{w_{ihj}(\hat{\delta})(y_{hj} - x_{hj}^{\mathrm{T}}\hat{\beta}_R - \hat{v}_{hR})\} + N_i^{-1} \sum_{h=1}^{k} \psi_{c_2}\{W_{ih}(\hat{\delta})\hat{v}_{hR}\}.
\end{aligned} \tag{16}$$

The robust predictor (16) is based on ψ functions, and we propose to use the Huber function given by (5), with appropriate choices of the tuning constants c_1 and c_2. When c_1 and c_2 converge toward zero, (16) tends towards the Sinha–Rao predictor, whereas it tends towards the empirical best linear unbiased predictor when the tuning constants converge to infinity. Unlike the robust predictor (11), the correction terms of $\hat{\theta}_{i\mathrm{C}}$ depend on s, not only s_i, which explains why the proposed predictor is called fully bias-corrected. The tuning constants need to be specified. They should be relatively large in order to lead to small biases, but small enough to ensure robustness and small mean square errors. The second and third terms of (16) are based on weighted residuals. A natural choice for c_1 is based on a robust estimator of the scale σ_e and a measure of the weight. Thus, $c_1 = q \times \mathrm{med}(w_{ihj}) \times \hat{\sigma}_{eR}$ with some constant q seems reasonable, where $\mathrm{med}(\cdot)$ denotes the median. If the outliers are non-representative, small values of q would be appropriate, for example $q = 3$ or even smaller values. In our framework, the outliers are representative, and thus a larger q is expected. From the simulation experiments in § 5, a value of q as large as $q = 9$ seemed to control the biases well, and the corresponding empirical mean square errors were small. Similarly, $c_2 = q \times \mathrm{med}(W_{ih}) \times \hat{\sigma}_{vR}$ seem natural for some value of q. See § 5.

3·2. *Robust prediction based on the conditional bias*

In predicting a population mean or total using auxiliary information and linear models, Beaumont et al. (2013) developed a model-based predictor using the conditional bias of a unit and showed that their proposal is closely related to the approach of Chambers (1986). This suggests that the conditional bias may be useful in the present framework.

Complications occur in mixed linear models, because the observations in a given area are correlated, due to area-specific random effects. Following Beaumont et al. (2013), we define the conditional bias of unit j in area h as

$$B_{ihj}(y_{hj}, v_h; \beta, \delta) = E_m\{\hat{\theta}_i(\delta) - \theta_i \mid s, y_{hj}, v_h\}. \tag{17}$$

Thus, in linear mixed models, we calculate the conditional expectations, keeping fixed a given unit and also the local effect associated with the area that contains that particular unit. Consequently, the conditional bias measures the average joint effect of unit y_{hj} and local area effect v_h, j in area h, on the predictor $\hat{\theta}_i$. We adopt definition (17) because of its simplicity, and in view of the relations with Chambers' approach discussed in § 3·1, as discussed below.

Several situations need to be considered, since a unit j may or may not be in the area of interest. Using the definition of the weights $w_{ihj}(\delta)$ in (12), we obtain

$$B_{ihj}(y_{hj}, v_h; \beta, \delta) = \begin{cases} N_i^{-1}(w_{iij} - 1)(y_{ij} - x_{ij}^{\mathrm{T}}\beta - v_i) + N_i^{-1} W_{ii} v_i & (j \in s_i), \\ N_i^{-1} w_{ihj}(y_{hj} - x_{hj}^{\mathrm{T}}\beta - v_h) + N_i^{-1} W_{ih} v_h & (j \in s_h, h \neq i), \\ -N_i^{-1}(y_{ij} - x_{ij}^{\mathrm{T}}\beta - v_i) + N_i^{-1} W_{ii} v_i & (j \in U_i - s_i), \\ N_i^{-1} W_{ih} v_h & (j \in U_h - s_h, h \neq i), \end{cases} \tag{18}$$

where $w_{ihj} \equiv w_{ihj}(\delta)$ and $W_{ih} \equiv W_{ih}(\delta)$. The result (18) suggests that a unit outside the area of interest may have a significant impact. In fact, a unit $j \in s_h$ may have a large influence if its weight $w_{ihj}(\delta)$ is large. Naturally, a large model residual of a given unit is expected to have a large influence, and a large residual $y_{hj} - x_{hj}^{\mathrm{T}}\beta - v_h$ is associated to a large conditional bias for unit j. Finally, a large random effect v_h increases the conditional bias, and the effect is more pronounced if the associated weight W_{ih} is large. Nonsampled units may have large influences, but it is not possible to reduce their impact at the estimation stage.

From expressions (2) and (18), the prediction error of the best linear unbiased predictor can be written as

$$\hat{\theta}_i(\delta) - \theta_i = \sum_{h=1}^{k} \sum_{j \in U_h} B_{ihj}(y_{hj}, v_h; \beta, \delta) - N_i^{-1} \sum_{h=1}^{k} (N_h - 1) W_{ih}(\delta) v_h. \qquad (19)$$

Expression (19) suggests that the conditional bias $B_{ihj}(y_{hj}, v_h; \beta, \delta)$ attached to unit j and the random effect v_h can be interpreted as their contribution to the prediction error of $\hat{\theta}_i(\delta)$. Following the approach of Beaumont et al. (2013), we define a robust pseudo-best linear unbiased predictor of θ_i,

$$\tilde{\theta}_i(\beta, \delta, v) = \theta_i + \sum_{h=1}^{k} \sum_{j \in s_h} \Phi_{d_1, d_2}\{B_{ihj}(y_{hj}, v_h; \beta, \delta)\} + \sum_{h=1}^{k} \sum_{j \in U_h - s_h} B_{ihj}(y_{hj}, v_h; \beta, \delta) - N_i^{-1} \sum_{h=1}^{k} (N_h - 1) \psi_{d_2}\{W_{ih}(\delta) v_h\}, \qquad (20)$$

where $v = (v_1, \ldots, v_k)^{\mathrm{T}}$ corresponds to the vector of random effects, and we define robustified conditional biases as

$$\Phi_{d_1, d_2}\{B_{ihj}(y_{hj}, v_h; \beta, \delta)\} = \begin{cases} N_i^{-1} \psi_{d_1}\{(w_{iij}(\delta) - 1)(y_{ij} - x_{ij}^{\mathrm{T}} \beta - v_i)\} + N_i^{-1} \psi_{d_2}\{W_{ii}(\delta) v_i\} & (j \in s_i), \\ N_i^{-1} \psi_{d_1}\{w_{ihj}(\delta)(y_{hj} - x_{hj}^{\mathrm{T}} \beta - v_h)\} + N_i^{-1} \psi_{d_2}\{W_{ih}(\delta) v_h\} & (j \in s_h, h \neq i), \end{cases} \qquad (21)$$

with the ψ functions in the Huber class, with some choices of tuning constants d_1 and d_2. Let $B_{ihj} \equiv B_{ihj}(y_{hj}, v_h; \beta, \delta)$. Using (19) and (20), it follows that

$$\tilde{\theta}_i(\beta, \delta, v) = \left\{ \hat{\theta}_i(\delta) - \sum_{h=1}^{k} \sum_{j \in U_h} B_{ihj} + N_i^{-1} \sum_{h=1}^{k} (N_h - 1) W_{ih}(\delta) v_h \right\} + \sum_{h=1}^{k} \sum_{j \in s_h} \Phi_{d_1, d_2}(B_{ihj}) + \sum_{h=1}^{k} \sum_{j \in U_h - s_h} B_{ihj} - N_i^{-1} \sum_{h=1}^{k} (N_h - 1) \psi_{d_2}\{W_{ih}(\delta) v_h\}. \qquad (22)$$

The first term on the right-hand side of (22) equals θ_i; thus, it is not affected by outliers. Interestingly, if $d_2 = \infty$, simplifications occur, since the weighted means of random effects cancel:

$$\tilde{\theta}_i(\beta, \delta, v) = \hat{\theta}_i(\delta) - \sum_{h=1}^{k} \sum_{j \in s_h} B_{ihj}(y_{hj}, v_h; \beta, \delta) + \sum_{h=1}^{k} \sum_{j \in s_h} \Phi_{d_1, \infty}\left\{B_{ihj}(y_{hj}, v_h; \beta, \delta)\right\}. \qquad (23)$$

The form of the pseudo-predictor (23) is identical to that proposed in Beaumont et al. (2013). In the following, we adopt $d_2 = \infty$. From (23), when a sample unit j has a small conditional bias, we have $\Phi_{d_1, \infty}\{B_{ihj}(y_{hj}, v_h; \beta, \delta)\} \approx B_{ihj}(y_{hj}, v_h; \beta, \delta)$ and the contribution of the second and third terms on the right-hand side of (23) is expected to be small. Thus, if the influences are small, the predictor (23) is close to the best linear unbiased predictor, the constant d_1 controlling the influences when they are significantly large.

The conditional biases $B_{ihj}(y_{hj}, v_h; \beta, \delta)$ are unknown and depend on the model parameters $(\beta^{\mathrm{T}}, \delta^{\mathrm{T}})^{\mathrm{T}}$ and also on the random small-area effects v_h. The vector of parameters for the fixed effects β and the random effects can be estimated with the methods of Sinha & Rao (2009), as described in § 2. Concerning the estimation of the variance components, a possible choice is to estimate δ by maximum likelihood; recall that in our framework the outliers are legitimate observations, and it appears desirable to have a predictor not too far from the empirical best linear unbiased predictor. Let $\hat{B}_{ihj} = B_{ihj}(y_{hj}, \hat{v}_{hR}; \hat{\beta}_R, \hat{\delta})$ denote the estimator of $B_{ihj}(y_{hj}, v_h; \beta, \delta)$ based on $\hat{\delta}$ and the robust estimators $\hat{\beta}_R$ and $\hat{v}_R = (\hat{v}_{1R}, \ldots, \hat{v}_{kR})^{\mathrm{T}}$. The following predictor represents a compromise between the empirical best linear unbiased predictor and the Sinha–Rao predictor:

$$\tilde{\theta}_i(\hat{\beta}_R, \hat{\delta}, \hat{v}_R) = \hat{\theta}_{i\text{EBLUP}} - \sum_{h=1}^{k} \sum_{j \in s_h} \hat{B}_{ihj} + \sum_{h=1}^{k} \sum_{j \in s_h} \Phi_{d_1,\infty}(\hat{B}_{ihj}), \tag{24}$$

where $\hat{\theta}_i(\hat{\delta}) = \hat{\theta}_{i\text{EBLUP}}$ in (24) is given by (3). When $d_1 = \infty$, the robust predictor (24) reduces to the empirical best linear unbiased predictor, which is asymptotically unbiased under model (1) but suffers from a potentially large variance in the presence of outliers. On the other hand, when d_1 converges toward zero, then (24) is expected to be highly robust but substantially biased. Using the definitions of $\Phi_{d_1,\infty}(\hat{B}_{ihj})$ given by (18) and (21) respectively, the robust predictor denoted by $\hat{\theta}_{i\text{CB}}$ simplifies to

$$\hat{\theta}_{i\text{CB}} = \hat{\theta}_{i\text{SR}} + N_i^{-1} \sum_{j \in s_i} \psi_{d_1}\{(w_{iij}(\hat{\delta}) - 1)(y_{ij} - x_{ij}^{\mathrm{T}}\hat{\beta}_R - \hat{v}_{iR})\}$$
$$+ N_i^{-1} \sum_{\substack{h=1 \\ h \neq i}}^{k} \sum_{j \in s_h} \psi_{d_1}\{w_{ihj}(\hat{\delta})(y_{hj} - x_{hj}^{\mathrm{T}}\hat{\beta}_R - \hat{v}_{hR})\} + N_i^{-1} \sum_{h=1}^{k} W_{ih}(\hat{\delta})\hat{v}_{hR}. \tag{25}$$

From (25), the robust predictor $\hat{\theta}_{i\text{CB}}$ is closely related to Chambers' approach as described in § 3·1. In fact, as $\hat{\theta}_{i\text{C}}$ defined by (16), the robust predictor $\hat{\theta}_{i\text{CB}}$ is fully bias-corrected, with correction terms based on robustified residuals for $j \in \cup_{h=1}^{k} s_h$ and random effects predictors $\hat{v}_{hR}$. When the tuning constant $d_1 \to \infty$, then (25) converges to the empirical best linear unbiased predictor $\hat{\theta}_{i\text{EBLUP}}$, and when $d_1 \to 0$ the predictor $\hat{\theta}_{i\text{CB}}$ tends to the Sinha–Rao predictor plus an additional term corresponding to a weighted mean of robust random effects predictors. In fact, (16) and (25) differ in the treatment of $N_i^{-1} \sum_{h=1}^{k} W_{ih}(\hat{\delta})\hat{v}_{hR}$, and large values of the weighted random effect $W_{ih}(\hat{\delta})\hat{v}_{hR}$ can be downweighted under Chambers' approach. The extra term comes from the empirical best linear unbiased predictor representation as the Sinha–Rao predictor plus correction terms, where expression (15) is used to obtain an explicit expression of (24) in terms of robust residuals and random effects. The simulation results presented in § 5 suggest that $\hat{\theta}_{i\text{C}}$ and $\hat{\theta}_{i\text{CB}}$ perform well, with $\hat{\theta}_{i\text{CB}}$ offering less variability across the domains than $\hat{\theta}_{i\text{C}}$, at least in our simulation experiments.

4. Asymptotic biases and mean square prediction error

4·1. *Asymptotic biases*

An asymptotic approach is adopted to find the limiting expected values of the prediction errors $\hat{\theta}_i - \theta_i$, when $\hat{\theta}_i$ is one of the robust predictors $\hat{\theta}_{i\text{SR}}$, $\hat{\theta}_{i\text{C}}$ and $\hat{\theta}_{i\text{CB}}$. Asymptotic properties of the robust estimators $\hat{\beta}_R$ and $\hat{\delta}_R$ have been studied in Sinha & Rao (2009), assuming that the populations are generated according to (1). The asymptotic arguments assumed that $k \to \infty$, with fixed values of n_i, which can be relatively small.

Let ζ_γ be a population from which the model for observations is given by

$$\zeta_\gamma : y_{\gamma i} = X_i \beta_\gamma + v_{\gamma i} 1_{n_i} + e_{\gamma i} \quad (i = 1, \ldots, k).$$

The underlying model for the values of Y, denoted by ζ_m, is assumed to be a mixture of ζ_0 and ζ_1, in the sense that the population values for Y in domain i are

$$y_{ij} = (1 - A_{ij}) y_{0ij} + A_{ij} y_{1ij} \quad (j \in U_i ; i = 1, \ldots, k), \tag{26}$$

where A_{ij} are independent realizations of Bernoulli random variables with parameter $p = \text{pr}(A_{ij} = 1)$. The model (1) is fitted. Let $\hat{\lambda}_m = (\hat{\beta}^{\mathrm{T}}, \hat{\delta}^{\mathrm{T}})^{\mathrm{T}}$ be the estimator under the working model. For example, $\hat{\beta}$ and $\hat{\delta}$ can be the empirical best linear unbiased estimator of β and the maximum likelihood estimator of δ assuming normality. The robust estimator $\hat{\lambda}_R = (\hat{\beta}_R^{\mathrm{T}}, \hat{\delta}_R^{\mathrm{T}})^{\mathrm{T}}$, solution of the robustified maximum likelihood equations (8) and (9), is calculated. Under standard regularity conditions, a first-order Taylor series expansion gives the following asymptotic expression for the bias under model ζ_m:

$$\begin{aligned}
\mathrm{AE}_m(\hat{\theta}_{i\mathrm{C}} - \theta_i) = &-N_i^{-1} \sum_{j \in U_i - s_i} [x_{ij}^{\mathrm{T}} (\beta_m - \beta_R) - E_m\{\hat{v}_{iR}(\lambda_R)\}] \\
&+ N_i^{-1} \sum_{j \in s_i} E_m(\psi_{c_1}[\{w_{iij}(\delta_m) - 1\}\{y_{ij} - x_{ij}^{\mathrm{T}} \beta_R - \hat{v}_{iR}(\lambda_R)\}]) \\
&+ N_i^{-1} \lim_{k \to \infty} \sum_{h \neq i} \sum_{j \in s_h} E_m(\psi_{c_1}[w_{ihj}(\delta_m)\{y_{hj} - x_{hj}^{\mathrm{T}} \beta_R - \hat{v}_{hR}(\lambda_R)\}]) \\
&+ N_i^{-1} \lim_{k \to \infty} \sum_{h=1}^{k} E_m[\psi_{c_2}\{W_{ih}(\delta_m) \hat{v}_{hR}(\lambda_R)\}],
\end{aligned} \tag{27}$$

where AE_m stands for the asymptotic expectation as $k \to \infty$ under the model ζ_m, the vector $\lambda_R = (\beta_R^{\mathrm{T}}, \delta_R^{\mathrm{T}})^{\mathrm{T}}$ is the probability limit of $\hat{\lambda}_R$, and $\lambda_m = (\beta_m^{\mathrm{T}}, \delta_m^{\mathrm{T}})^{\mathrm{T}}$ is the probability limit of $\hat{\lambda}_m$, with $\beta_m = (1 - p)\beta_0 + p\beta_1$. The main lines of the proof are given in the Supplementary Material. When the tuning constant $c_1 = d_1$ and if c_2 converges to infinity, we obtain the asymptotic bias of the predictor (25). The asymptotic bias of the Sinha–Rao predictor is obtained by letting c_1 and c_2 converge to zero in (27):

$$\mathrm{AE}_m(\hat{\theta}_{i\mathrm{SR}} - \theta_i) = (1 - n_i N_i^{-1}) \, [\bar{X}_{\bar{s}_i}^{\mathrm{T}} (\beta_R - \beta_m) + E_m\{\hat{v}_{iR}(\lambda_R)\}], \tag{28}$$

where $\bar{X}_{i\bar{s}_i} = (N_i - n_i)^{-1} \sum_{j \in U_i - s_i} x_{ij}$. The additional terms in (27) represent the correction terms for the bias. When $c_1, c_2 \to \infty$ in (27) the calibration constraints (13) show that the asymptotic biases are zero, as expected. Choosing small values of c_1 and c_2 introduces some bias, but the gains in mean squared error are typically significant, see § 5. The asymptotic bias may be appreciable for the Sinha–Rao method if the limit of the robust estimator β_R is different from the parameter β_m of the mixture model. In model (26) with different slopes and no contamination in the random effects, the main term in (28) is $\beta_m - \beta_R$. In the limit case $\sigma_v^2 = 0$, the bias depends only on $\beta_m - \beta_R$. On the other hand, when the fixed effects in the mixture model have the same slopes, that is $\beta_0 = \beta_1$, and if the random effects v_h represent mixtures such as the contaminated normal distributions $(1-p)\mathcal{N}(0,\sigma_{v0}^2)+p\mathcal{N}(0,\sigma_{v1}^2)$, the biases of the Sinha–Rao predictor are also expected to be small. In the latter situation, the contaminations affect the variance, not the random effects means, and both terms in (28) will be small. Incidentally, this scenario has been studied in Sinha & Rao (2009), who found good behaviour for their robust predictor. Expression (28) suggests that the mixture model with $\beta_0 \neq \beta_1$ or contamination in the random effects with nonnull mean may create nonnegligible biases for the Sinha–Rao predictor.

4·2. *Estimation of the mean square prediction error*

Constructing confidence intervals for the robust predictors is important but challenging. Sinha & Rao (2009) proposed a parametric bootstrap procedure to estimate the mean square prediction error but it gave poor coverage rates in our empirical results for mixture models, because the populations were generated using the robust estimators, which do not reflect all the units. In our framework, all the observations are legitimate and using robust estimators reflected only the main component of the mixtures resulting in large biases. Let $\hat{\beta}_R$ be the robust estimator of β, and $\hat{\delta}$ be the nonrobust estimator of δ. For generating the bootstrap populations, a reasonable solution relies on using the robust estimators for the slope parameters, and $\hat{\delta}$ for estimating δ, which rely on all the legitimate observations defining the population. This comes from the form of the predictors (15) and (24), which are functions of the estimator $\hat{\delta}$ and also of the robust residuals. We present the method for $\hat{\theta}_{i\mathrm{C}}$, but it can be easily adapted for other predictors.

Step 1. The random variables v_i^* and e_{ij}^* are generated according to $\mathcal{N}(0,\hat{\sigma}_v^2)$ and $\mathcal{N}(0,\hat{\sigma}_e^2)$, respectively, to create bootstrap samples from the model

$$y_{ij}^* = x_{ij}^{\mathrm{T}}\hat{\beta}_R + v_i^* + e_{ij}^*.$$

Step 2. Let $\mathcal{E}_i^*$ be the mean error of the nonsampled units from normal distributions $\mathcal{N}\{0,(N_i - n_i)^{-1}\hat{\sigma}_e^2\}$. The bootstrap mean θ_i^* is defined as

$$\theta_i^* = N_i^{-1}\sum_{j\in s_i} y_{ij}^* + N_i^{-1}\sum_{j\in U_i - s_i} x_{ij}^{\mathrm{T}}\hat{\beta}_R + (1-n_iN_i^{-1})v_i^* + (1-n_iN_i^{-1})\mathcal{E}_i^*. \tag{29}$$

Step 3. From the bootstrap samples, robust bootstrap estimators $\hat{\beta}_R^*$, $\hat{\delta}_R^*$, $\hat{v}_{hR}^*$ and estimators $\hat{\beta}^*$, $\hat{\delta}^*$, $\hat{v}_h^*$ of β, δ, v_h are calculated. The parametric bootstrap estimator of the small-area mean is

$$\begin{aligned}\hat{\theta}_{i\mathrm{C}}^* = \hat{\theta}_{i\mathrm{SR}}^* &+ N_i^{-1}\sum_{j\in s_i}\psi_{c_1}[\{w_{ij}(\hat{\delta}^*)-1\}(y_{ij}-x_{ij}^{\mathrm{T}}\hat{\beta}_R^* - \hat{v}_{iR}^*)] \\ &+ N_i^{-1}\sum_{\substack{h=1\\h\neq i}}^{k}\sum_{j\in s_h}\psi_{c_1}\{w_{ihj}(\hat{\delta}^*)(y_{hj}-x_{hj}^{\mathrm{T}}\hat{\beta}_R^*-\hat{v}_{hR}^*)\} \\ &+ N_i^{-1}\sum_{h=1}^{k}\psi_{c_2}\{W_{ih}(\hat{\delta}^*)\hat{v}_{hR}^*\},\end{aligned} \tag{30}$$

where $\hat{\theta}_{i\mathrm{SR}}^* = N_i^{-1}\sum_{j\in s_i} y_{ij}^* + N_i^{-1}\sum_{j\in U_i-s_i} x_{ij}^{\mathrm{T}}\hat{\beta}_R^* + (1-n_iN_i^{-1})\hat{v}_{iR}^*$.

Step 4. Based on B bootstrap samples, the estimator of the mean square prediction error of $\hat{\theta}_{i\text{C}}$ is

$$\text{MSE}(\hat{\theta}_{i\text{C}}) = B^{-1} \sum_{b=1}^{B} \left(\hat{\theta}_{i\text{C}}^{*(b)} - \theta_i^{*(b)} \right)^2,$$

where $\theta_i^{*(b)}$ and $\hat{\theta}_{i\text{C}}^{*(b)}$ correspond to (29) and (30), respectively, for the bth bootstrap sample.

Depending on the nature of the outliers, the proposed bootstrap method is expected to work reasonably well. Examples include outliers in the random effects and outliers in the error term. When the slope parameters are different in the mixture, the proposed bootstrap method is expected to be biased. From the results in § 5, the performance of the proposed method was encouraging under all scenarios, at least in our experiments, see § 5.

5. Monte Carlo experiments

5·1. *Description of the populations*

We conducted a Monte Carlo study in order to assess the empirical biases and mean square errors of the new robust predictors $\hat{\theta}_{i\text{C}}$ and $\hat{\theta}_{i\text{CB}}$, and to compare them to the proposals of Sinha & Rao (2009) and Chambers et al. (2013). We also study the performance of the bootstrap method. In view of the results in § 4, to compare empirically the biases of $\hat{\theta}_{i\text{SR}}$ with those of the new methods seem particularly relevant. We considered the mixture model ζ_m of two unit-level models:

$$\zeta_0 : y_{0ij} = \beta_{00} + \beta_{01} x_{ij} + v_{0i} + e_{0ij} \quad (j = 1, \ldots, N_i; i = 1, \ldots, k), \tag{31}$$

$$\zeta_1 : y_{1ij} = \beta_{10} + \beta_{11} x_{ij} + v_{1i} + e_{1ij} \quad (j = 1, \ldots, N_i; i = 1, \ldots, k). \tag{32}$$

We considered $k = 40$ and $N_1 = \cdots = N_{40} = 50$. Normal distributions for the random effects and the error terms were assumed. Thus, $v_{0i} \sim \mathcal{N}(0, \sigma_{v0}^2)$, $v_{1i} \sim \mathcal{N}(0, \sigma_{v1}^2)$, $e_{0ij} \sim \mathcal{N}(0, \sigma_{e0}^2)$ and $e_{1ij} \sim \mathcal{N}(0, \sigma_{e1}^2)$ $(k = 1, \ldots, 40;\ j = 1, \ldots, 50)$. The values of the auxiliary information were generated from a normal distribution such that $E(X) = 2$ and $\text{var}^{1/2}(X) = 0{\cdot}35$. The mixture model ζ_m satisfied $y_{ij} = (1 - A_{ij}) y_{0ij} + A_{ij} y_{1ij}$, where the A_{ij} were independently generated according to a Bernoulli distribution, $A_{ij} \sim \text{Ber}(p)$, with $p = 0{\cdot}1$. In each area of a given population, random samples of size $n_1 = \cdots = n_{40} = 5$ have been selected by simple random sampling without replacement.

Contamination in the error terms and random effects was investigated in the simulation experiments of Sinha & Rao (2009). More precisely, the random effects were generated according to the contaminated normal distribution $(1 - p)\mathcal{N}(0, \sigma_{v0}^2) + p\mathcal{N}(0, \sigma_{v1}^2)$. Similarly, the error terms were obtained from $(1 - p)\mathcal{N}(0, \sigma_{e0}^2) + p\mathcal{N}(0, \sigma_{e1}^2)$. In view of the asymptotic biases derived in § 4, it is particularly relevant to investigate situations where the intercepts and slopes are different. We considered the same scenarios as in Sinha & Rao (2009), but added two more scenarios with $\beta_0 \neq \beta_1$, where $\beta_0 = (\beta_{00}, \beta_{01})^{\text{T}}$ and $\beta_1 = (\beta_{10}, \beta_{11})^{\text{T}}$. Adopting a notation similar to that of Sinha & Rao (2009), three scenarios were studied, see Table 1.

From Table 1, scenario (0, 0, 0) corresponds to the absence of contamination. For the parameters in Table 1 the correlation between the units of a given area is set to $\rho_0 = 0{\cdot}5$ in the absence of contamination. Note that ρ_0 satisfies the relation $\sigma_{v0}^2 = \rho_0 \sigma_{e0}^2 / (1 - \rho_0)$. We also considered the case $\rho_0 = 0{\cdot}05$ in the Supplementary Material. Under $(0, 0, b)$, the random effects and the error terms are not contaminated, but the model parameters in (31) and (32) are different. Under $(0, v, 0)$, only the area random effect is contaminated. Another example is the case (e, v, b),

Table 1. *Description of the three scenarios. The populations were generated according to* $y_{ij} = (1 - A_{ij})y_{0ij} + A_{ij}y_{1ij}$, $A_{ij} \sim \text{Ber}(0{\cdot}1)$*, using the unit-level models* (31) *and* (32)*, assuming normality for the random effects and error terms in* ζ_0 *and* ζ_1*. Under the scenario* $(0, 0, 0)$*, the correlation between the units of the same domain equals* $0{\cdot}5$

	Sources of contamination		
Scenarios	Variances (error terms)	Variances (random effects)	Intercepts and slopes
$(0, 0, 0)$	$(\sigma^2_{e0}, \sigma^2_{e1}) = (6, 6)$	$(\sigma^2_{v0}, \sigma^2_{v1}) = (6, 6)$	$\beta_0 = \beta_1 = (100, 3)^{\mathrm{T}}$
$(e, v, 0)$	$(\sigma^2_{e0}, \sigma^2_{e1}) = (6, 150)$	$(\sigma^2_{v0}, \sigma^2_{v1}) = (6, 150)$	$\beta_0 = \beta_1 = (100, 3)^{\mathrm{T}}$
(e, v, b)	$(\sigma^2_{e0}, \sigma^2_{e1}) = (6, 150)$	$(\sigma^2_{v0}, \sigma^2_{v1}) = (6, 150)$	$\beta_0 = (100, 3)^{\mathrm{T}}, \beta_1 = (150, 1)^{\mathrm{T}}$

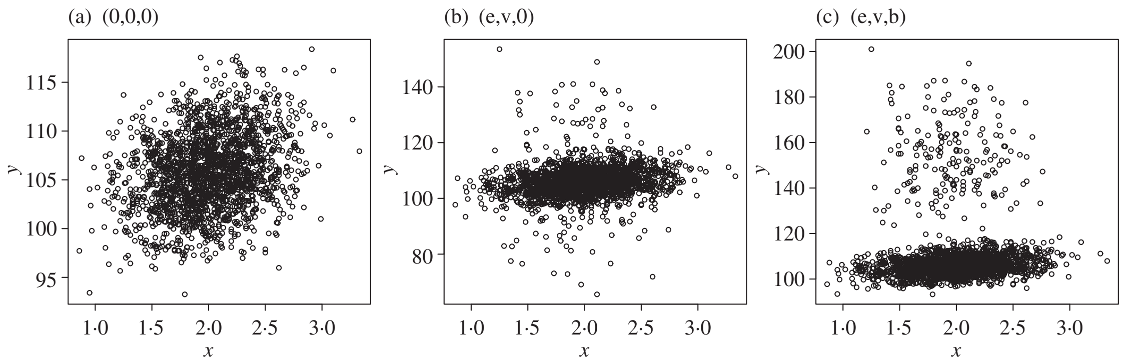

Fig. 1. Populations generated according to the mixture model (31) and (32). The model parameters are given in Table 1. Under the scenario $(0, 0, 0)$, the correlation between the units of the same domain is set to $0{\cdot}5$.

where the contamination comes from the random effects, the random errors and the fixed effects. The other scenarios are interpreted in the same manner. See Fig. 1.

5·2. *Predictors used in the study and empirical measures*

Five predictors of the small-area mean θ_i were included in the study. The empirical best linear unbiased predictor defined by (3), based on the empirical best linear unbiased estimators and empirical best linear unbiased predictors, with variance components estimated by maximum likelihood, was used as a benchmark. As in Sinha & Rao (2009), the robust predictors relied on the robustified maximum likelihood estimators of $(\beta^{\mathrm{T}}, \delta^{\mathrm{T}})^{\mathrm{T}}$, which are obtained by solving (8) and (9). The robust random effects were estimated by solving Fellner's equation (4). We used $b = 1{\cdot}345$. Based on these estimators, the following robust predictors were considered: the Sinha–Rao predictor $\hat{\theta}_{i\mathrm{SR}}$ given by (10); the new robust procedures $\hat{\theta}_{i\mathrm{C}}$ and $\hat{\theta}_{i\mathrm{CB}}$ defined by (16) and (25), respectively; and the predictor (11), denoted by $\hat{\theta}_{i\mathrm{CCST}}$. For $\hat{\theta}_{i\mathrm{C}}$, the tuning constants were set to $c_1 = q \times \mathrm{med}\{w_{ihj}(\hat{\delta})\} \times \hat{\sigma}_{eR}$ and $c_2 = q \times \mathrm{med}\{W_{ih}(\hat{\delta})\} \times \hat{\sigma}_{vR}$, where $(\hat{\sigma}^2_{eR}, \hat{\sigma}^2_{vR})^{\mathrm{T}}$ denote the robust estimators of the variance components based on the robustified maximum likelihood method. We considered $d_1 = c_1$ in $\hat{\theta}_{i\mathrm{CB}}$. In § 5·3, simulation results are presented for all the domains using boxplots, and we considered in these cases $q = 3, 6, 9$ for the predictors $\hat{\theta}_{i\mathrm{C}}$ and $\hat{\theta}_{i\mathrm{CB}}$. The tuning constant of the robust predictor $\hat{\theta}_{i\mathrm{CCST}}$ was set to $c = 1, 2, 3$, inspired by the simulation experiments of Chambers et al. (2013).

For each scenario described in Table 1, 1000 populations were generated. Let $\hat{\theta}_i^{(b)}$ denote a predictor for domain i at iteration b. The empirical percent absolute relative bias for the area

Table 2. *Monte Carlo relative biases* (%) *for the predictors of the small-area means averaged over areas*

	EBLUP	CBb	CB3	CB6	CB9	Cb	C3	C6	C9	SR	CCST1	CCST2	CCST3
$(0, 0, 0)$	0·03	0·03	0·02	0·03	0·03	0·03	0·03	0·03	0·03	0·03	0·03	0·03	0·03
$(e, v, 0)$	0·05	0·04	0·04	0·05	0·05	0·03	0·04	0·04	0·04	0·04	0·04	0·04	0·04
(e, v, b)	0·07	1·80	1·08	0·55	0·28	1·84	1·11	0·58	0·31	3·14	3·06	2·80	2·52

The robust predictor $\hat{\theta}_{i\text{C}}$ is denoted Cb, C3, C6 and C9, when the tuning constants are set to $q = b$, 3, 6, 9, respectively. Similarly, $\hat{\theta}_{i\text{CB}}$ is represented by CBb, CB3, CB6 and CB9. The Sinha–Rao predictor is denoted SR, and the predictor $\hat{\theta}_{i\text{CCST}}$ is denoted CCST1, CCST2 and CCST3, when the tuning constants are $c = 1, 2, 3$, respectively. Under the scenario (0, 0, 0), the correlation between the units of the same domain equals 0·5.

mean θ_i was calculated as

$$\text{ARB}(\hat{\theta}_i) = \left| B^{-1} \sum_{b=1}^{B} \frac{\left(\hat{\theta}_i^{(b)} - \theta_i^{(b)}\right)}{\theta_i^{(b)}} \right| \times 100 \quad (i = 1, \ldots, k).$$

Using the empirical best linear unbiased predictor $\hat{\theta}_{i\text{EBLUP}}$ as the reference, we calculated the relative efficiency of $\hat{\theta}_i$ in percentage $\hat{\theta}_{i\text{EBLUP}}$, using

$$\text{RE}(\hat{\theta}_i) = \frac{\text{MSE}(\hat{\theta}_i)}{\text{MSE}(\hat{\theta}_{i\text{EBLUP}})} \times 100, \quad \text{MSE}(\hat{\theta}_i) = B^{-1} \sum_{b=1}^{B} \left(\hat{\theta}_i^{(b)} - \theta_i^{(b)}\right)^2 \quad (i = 1, \ldots, k),$$

with $\text{MSE}(\hat{\theta}_i)$ denoting the empirical mean squared error of $\hat{\theta}_i$. Section 4·3 presents simulation results based on all the domains, using boxplots and integrated measures such as averages computed over all the areas.

We also present boxplots for 95% confidence intervals using the bootstrap method. We used 500 populations and $B = 200$ bootstrap replications. In addition to the empirical best linear unbiased predictor and the proposed methods, we considered $\hat{\theta}_{i\text{CCST}}$ with $c = 1, 2$, which offered the best efficiencies. Simulation results are reported under scenarios $(0, v, 0)$, $(e, v, 0)$ and (e, v, b).

5·3. *Results for all the domains*

In Table 2, integrated measures of the empirical biases are given, where the averages over the areas were computed. Boxplots displaying empirical efficiency for all the domains are presented in Fig. 2, for all the scenarios described in Table 1. The relative efficiencies were computed for each domain in the Monte Carlo studies; boxplots of the $k = 40$ relative empirical efficiencies with respect to the empirical best linear unbiased predictor were computed for each domain.

Table 2 shows that all the methods exhibited small empirical biases for the scenarios (0, 0, 0), and $(e, v, 0)$. Larger biases were observed under scenario (e, v, b) for the robust methods as a function of the tuning constants. Small values of the tuning constants generated larger empirical biases. The empirical biases decreased as the tuning constants increased, as expected. From Table 2, all the empirical biases were small for these scenarios. From Fig. 2, the new robust methods were significantly more efficient than the robust predictors $\hat{\theta}_{i\text{SR}}$ or $\hat{\theta}_{i\text{CCST}}$, when the population was generated according to the mixture model.

Under scenario (0, 0, 0), the robust predictors $\hat{\theta}_{i\text{CB}}$ and $\hat{\theta}_{i\text{SR}}$ were as efficient as the empirical best linear unbiased predictor, whereas $\hat{\theta}_{i\text{C}}$ was slightly less efficient. The robust predictor $\hat{\theta}_{i\text{CCST}}$ was approximately unbiased but the variance part of the mean squared error was large. From

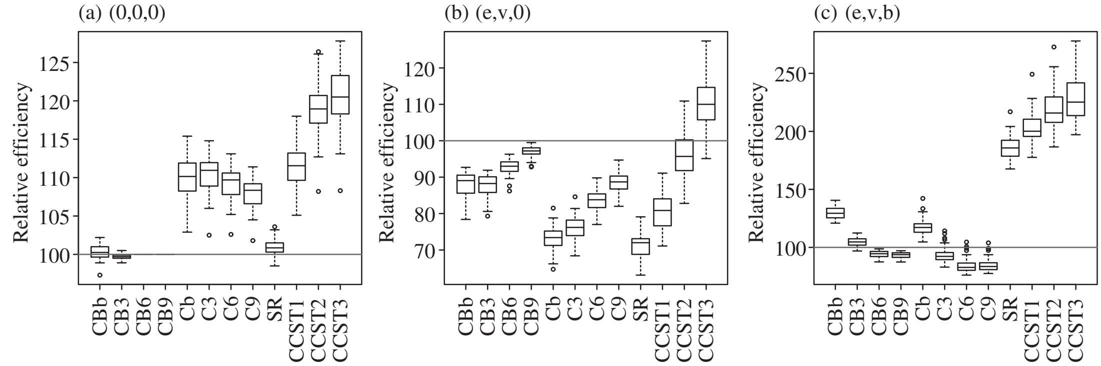

Fig. 2. Boxplots of the relative efficiencies for the predictors defined in Table 2. Under scenario (0, 0, 0), the correlation between the units of the same domain equals 0·5.

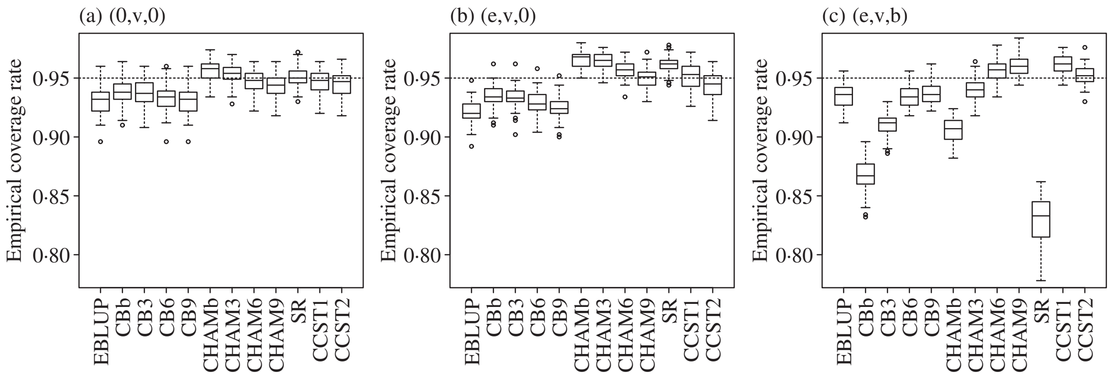

Fig. 3. Empirical coverage rates under scenarios $(0, v, 0)$, $(e, v, 0)$ and (e, v, b). The nominal coverage rate is 95%. Predictors are defined in Table 2. Under scenario (0, 0, 0), the correlation between the units of the same domain equals 0·5.

Fig. 2, the integrated mean square errors were 10–20% larger than the empirical best linear unbiased predictor. Under scenario (e, v, b), small differences in efficiency were observed between $\hat{\theta}_{i\text{C}}$ and $\hat{\theta}_{i\text{CB}}$, for a given value of the tuning constant. The predictors $\hat{\theta}_{i\text{SR}}$ and $\hat{\theta}_{i\text{CCST}}$ showed large mean square errors; for these predictors, the bias part of the mean squared error was nonnegligible. Efficiency gains were possible for several areas, and the medians of the boxplots suggest that the proposed robust methods were more efficient. Large gains in efficiency were observed under the scenario $(e, v, 0)$. As expected, the efficiencies decreased as the tuning constant increased. Taking $q = 3$ or $q = 6$ seemed to give robustness and efficiency. From Table 2, possible reductions in integrated mean square errors of approximately 10%–20% were observed for these values of the tuning constants.

Figure 3 presents the empirical coverage rates of the bootstrap confidence intervals. Due to the nature of the mixture models, the empirical coverage rates were reasonably close to the nominal coverage rates under $(0, v, 0)$ and $(e, v, 0)$. When the slopes were different, the bootstrap method worked reasonably well for the new methods with moderate tuning constants. The results for $\hat{\theta}_{i\text{CCST}}$ were reasonable. The biases were large for the Sinha–Rao method and the coverage rates were far from the nominal levels. Under $(0, v, 0)$, the biases of the mean squared error estimators were reasonably small, under 10%, for the proposed methods with moderate values of the tuning constants. These biases were small under $(e, v, 0)$ for the predictors based on the concept of conditional bias but large and positive using Chambers' method, around 10–20%. Under (e, v, b), the biases of the mean squared error estimators were positive for the new predictors with moderate values of the tuning constants, but large and negative for the Sinha–Rao method.

SUPPLEMENTARY MATERIAL

Supplementary material available at *Biometrika* online includes an outline of the proof of the asymptotic expression for the bias and additional simulation experiments.

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